# Numerical Solution of the Thermally Isolated Crack Problem Using Singular Integral Equations

**Gela Matcharashvili**

*Technical University of Georgia, M.Kostava st 77, Tbilisi 0159, Georgia.*

Macharashviligela22@gtu.ge

**DOI:** https://doi.org/10.52340/building.2025.72.02.12

**Abstract** The paper discusses the numerical solution of the thermally isolated crack problem using singular integral equations. A circular crack is given in an infinite body, on which the temperature distribution function is known. Determination of the intensity coefficients for stress distribution in a small region near the crack tips is based on a first-kind singular integral equation, which is solved using Markov-type quadrature formulas. The values of stresses in the vicinity of the crack are determined by Cauchy-type integrals; for their computation Professor M. Kublashvili's quadratic formula is used. A specific test problem is discussed. The program "Mathematika" has been compiled in a symbolic language.

**Keywords:** Thermal Insulation Crack, Singular Integral, Numerical Solution, Harmonic function, Algorithm.

## Introduction

The real strength of solid bodies essentially depends on real structural defects. In real materials, there always exist numerous microdefects of various types; under the action of an applied load, their growth in the direction of defect extension gives rise to cracks, which ultimately result in local or complete fracture of the body. As experience shows, this phenomenon is especially characteristic of brittle or quasi-brittle deformable solids.

The study of strength issues for structural elements and cracked structures is of great interest to many well-known researchers.

Many monographs are devoted to the theory of brittle fracture of solids: N. Morozov [1], V. Panasyuk, S. Savruk, A. Datsyshyn [2], V. Panasyuk [3], G. Cherepanov [4], N. Muskhelishvili [5], L. Sedov [6]. A broad review of these works is presented in the monograph [6]. In these works, one of the most important lines of research is accounting for the redistribution of stresses in bodies when cracks and holes arise in them.

## Main body

In the present work, numerical solutions of crack problems are studied using a singular integral equation. In particular, the numerical solution of the thermally isolated crack problem is considered.

Let an infinite body contain a thermally isolated crack along a segment. Assume that, in the body without the crack, the temperature distribution is described by a given harmonic function. Then the total temperature can be represented in the following form:

$$T(x,y)=t_0(x,y)+t(x,y), \tag{1}$$

where is the disturbed temperature field caused by the crack. Since the crack is thermally isolated, the following condition holds on its boundary [2]:

$$\frac{\partial T^+}{\partial y}=\frac{\partial T}{\partial y}=0,\quad |x|<a, y=0 \tag{2}$$

which, taking (1) into account, can be written as:

$$\frac{\partial t^+}{\partial y}=\frac{\partial t^-}{\partial y}=-\frac{\partial t_0}{\partial y}=\varphi(x),|x|<a, y=0 \tag{3}$$

From the condition of temperature continuity at the crack tips it follows that:

$$t^+(x,0)=t^-(x,0),$$

when $x = \pm a.$ (4)

Thus, the problem reduces to determining a harmonic function that vanishes at infinity and satisfies conditions (3) and (4).

Represent the disturbed temperature in the following form:

$$t(x, y) = \operatorname{Re} f(z), \quad (5)$$

where $f(z)$ is any piecewise holomorphic function. Then:

$$\frac{\partial t}{\partial x} = \operatorname{Re} F(z); \frac{\partial t}{\partial y} = -\operatorname{Im} F(z), \quad F(z) = f'(z)$$

(6)

Let us also introduce the distribution function. Therefore, the stress distribution in a small neighborhood of the crack tip will be known if the stress intensity coefficient is determined. Hence, determining the intensity coefficients $k_1^{\pm}$ and $k_2^{\pm}$ is of essential importance:

$$\psi(x) = \frac{1}{2}\left[t^{+}(x,0) - t^{-}(x,0)\right], \quad |x| < a,$$

(7)

which describes the temperature jump when crossing the crack line. As is known [2], for $\psi(x)$ one obtains a first-kind singular integral equation:

$$\frac{1}{\pi}\int_{-a}^{+a} \frac{\psi'(t)dt}{t-x} = \varphi(x), \quad |x| < a \quad (8)$$

with the additional condition:

We can obtain a numerical solution of the boundary-value problem (8) and (9) using the algorithm constructed by Zurab Kapanadze:

$$\int_{-1}^{+1} \rho(t)\frac{\phi(t)-\phi(x)}{t-x}dt \approx \sum_{k=1}^{n} A_k \frac{\phi(t_k)-\phi(x)}{t_k - x} (-1 < x < 1)$$

(10)

It is implied that $\varphi(t)$ possesses the corresponding smoothness; moreover, as $t_k = x$, in expression (10) the corresponding limit is understood. When $\varphi(t)$ is a $\le 2n$ polynomial, it is clear that for any $x \in (-1;\ +1)$ the given quadrature formula is exact; and if the values in (10) are chosen so that the equality holds:

$$\int_{-1}^{+1} \rho(t)\frac{dt}{t-x} - \sum_{k=1}^{n} \frac{A_k}{t_k - x} = 0,$$

(11)

then, for the corresponding values of $x$ (by condition (11)), for the following singular integral:

$$\int_{-1}^{+1} \rho(t)\frac{\phi(t)}{t-x}dt(-1 < x < 1)$$

we obtain the quadrature formula:

$$\int_{-1}^{+1} \rho(t)\frac{\varphi(t)}{t-x}dt \approx \sum_{k=1}^{n} A_k \frac{\varphi(t_k)}{t_k - x}(-1 < x < 1),$$

(12)

Quadrature formulas of this structure, known in the literature as Gaussian quadrature formulas for singular integrals, are determined completely unambiguously for a given $\rho(t)$ by the values of the $\{x_k\}_{k=1}^{n-1}$ singularity points. From this it follows that the number of values of the singularity parameter $x$ for which the highest algebraic degree of accuracy of formulas of the form (12) is achieved is strictly limited.

In this regard, considering the practical effectiveness of quadrature formulas that have a relatively high degree of accuracy, it is natural to attempt, to some extent, to increase the number of such singularity points $x$. In relation to formulas of the type (12), this may be realized at the expense of some reduction in accuracy, but in such a way that the resulting quadrature formulas still have a significantly higher accuracy than the interpolation-accuracy formulas that are used much more often (or formulas close to them). As numerical experiments show, as the value of $r$ is gradually decreased within given bounds, we obtain quadrature formulas with somewhat reduced, yet still near-Gaussian accuracy.

In constructing the most effective high-accuracy formulas for singular integrals of the form (12), computational experiments play a very essential role. In this context, it should once again be noted that the known effectiveness of using increased-accuracy quadrature formulas for singular integrals is confirmed by the numerical solution of problems related to solving integral equations containing Cauchy principal value singular integrals.

In (6), the function $F(z)$ has the following form:

$$F(z) = \frac{F^*(z)}{\sqrt{z^2 - a^2}} = \frac{1}{\pi i \sqrt{z^2 - a^2}} \int_{-a}^{+a} \frac{\sqrt{a^2 - t^2}\,\varphi(t)dt}{t - z}, \qquad (13)$$

where:

$$F^*(z) = \frac{1}{\pi \mathrm{i}} \int_{-a}^{+a} \frac{\sqrt{a^2 - t^2}\,\varphi(t)dt}{t - z}$$

Knowing what the function $F(z)$ is, from relation (6) we determine the disturbed temperature $t(x, y)$.

For a fixed point $z$, we can compute the integral by an ordinary quadrature formula, but the order of accuracy decreases substantially as the point approaches the boundary points of $[-a, +a]$ (especially when $z \to \pm a$).

We compute the $F^*(z)$ integral using the quadratic formula constructed by Professor Murman Kublashvili [7]:

$$F^*(z) \approx F_n^*(z) = F^*(1;z)L_v(\varphi;t_0) + p_{v-10}(t_0;z)\frac{\varphi(\tau_{v-1}) - \varphi(t_0)}{\tau_{v-1} - t_0} + [p_{v-11}(t_0,z) + p_{v0}(t_0,z) + p_{V1}(t_0,z) +$$

$$+ p_{v+10}(t_0,z)]\frac{\varphi(\tau_{v+1}) - \varphi(\tau_v)}{\tau_{v+1} - \tau_v} + p_{v+11}(t_0,z)\frac{\varphi(\tau_{v+2}) - \varphi(t_0)}{\tau_{v+2} - t_0} + \sum_{\sigma \neq v \pm 1}^{n} \sum_{k=0}^{e} p_{\sigma k}(t_0,z)\frac{\varphi(\tau_{v+1}) - \varphi(t_0)}{\tau_{v+k} - t_0},$$

where:

$$p_{\sigma k}(t_0, z) = \frac{1}{\pi i} \int_{\tau_\sigma \tau_{\sigma+1}} \frac{\sqrt{(t-a)(b-t)}\,(t - t_0)}{t - z} l_{\sigma k}(t)dt$$

($\sigma = 1, 2, \cdots, n$).

Table 1 presents an approximate computation table for the integral $F(z)$ using the algorithm given above, for different values of in the neighborhood of the point $z = \pm 1$, when $\varphi(t) = \mathrm{Re}(t) + \mathrm{Im}(t)$.

**Table 1**

| $n$ | $z$ | $F(z)$ | $F_n(z)$ | $R_n(z)=F(z)-F_n(z)$ |
|---|---|---|---|---|
| 10 | $-1+0.005i$ | $3.60173+3.46915i$ | $3.8148+3.66829i$ | $0.21307+0.19914i$ |
| 10 | $1+0.005i$ | $-3.60173+3.46915i$ | $-3.81848+3.66829i$ | $-0.21307+0.19914i$ |
| 20 | $-1+0.005i$ | $3.60173+3.46915i$ | $-3.69248+3.52953i$ | $0.09075+0.06038i$ |
| 20 | $1+0.005i$ | $-3.60173+3.46915i$ | $-3.69248+3.52953i$ | $-0.09075+0.06038i$ |
| 30 | $-1+0.005i$ | $3.60173+3.46915i$ | $3.69248+3.52953i$ | $0.04035+0.09338i$ |
| 30 | $1+0.005i$ | $3.60173+3.46915i$ | $-3.69248+3.52953i$ | $-0.04035+0.09338i$ |
| 50 | $-1+0.005i$ | $3.60173+3.46915i$ | $3.63742+3.45305i$ | $0.03569-0.01610i$ |
| 50 | $1+0.005i$ | $-3.60173+3.46915i$ | $-3.63742+3.45305i$ | $-0.03569-0.01610i$ |

Thus, as can be seen from the table, by means of the above algorithm $z=\pm 1+0{,}005i$, at special points (the crack ends) for $n=50$an accuracy order of $10^{-2}$ is achieved.

### Conclusion

Thus, the considered thermally insulated crack problem is reduced to a first-kind singular integral equation, in which the singular integral is replased by a numerical calculation scheme developed by us , distinguished by the simplicity of calculations and the relatively high accuracy of the obtained results (Gaussian- type accuracy) as confirmed by the results presented in the table.

**References**


1. Morozov, N.F. *Mathematical Questions of Crack Theory.* Moscow: "Nauka", 1984, 255 p.
2. Panasyuk, V.V., Savruk, M.P., Datsyshyn, A.P. *Stress Distribution Near Cracks in Plates and Shells.* Kyiv: Naukova Dumka, 1976, 443 p.
3. Panasyuk, V.V. *Limit Equilibrium of Brittle Bodies with Cracks.* Kyiv: "Naukova Dumka", 1968.
4. Cherepanov, G.P. *Mechanics of Brittle Fracture.* Moscow: "Nauka", 1974.
5. Muskhelishvili, N.I. *Some Basic Problems of the Mathematical Theory of Elasticity.* Moscow: Nauka, 1966, 707 p.
6. Sedov, L.I. *Continuum Mechanics,* Vol. 2. Moscow: "Nauka", 1973.
7. M. Kublashvili. *Development of Methods for Studying the Stress–Strain State and Fracture Processes of Bodies with Cracks Using the Method of Singular Integral Equations,* Doctoral Dissertation, Tbilisi, 2004.