ON THE ANALYSIS OF A TUNNEL LINING OF OVAL CROSS-SECTION BY THE VARIATION METHOD

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Summary A tunnel support (cylindrical shell) of oval cross-section, whose thickness and radius vary stepwise, is analyzed using the variational method.
A discontinuous solution of the governing differential equation is constructed based on the theory developed by Sh. E. Mikeladze.

Keywords: cylindrical shell, spherical shell, differential equation, variational method, discontinuous solution

Introduction

The development of engineering technology has consistently created the need for lightweight structures with high strength and stiffness. One such class of structures is **cylindrical shells**. Owing to geometric diversity and the nature of applied loads, displacements and deformations in these shells are proportional to the shell thickness. This necessitates the use of nonlinear relationships between deformations and displacements, the calculation of which relies on various assumptions and simplifications.
The efficiency of thin-walled shells is associated with the development of improved computational models and refinements of existing analytical methods. In the design of cylindrical shells, anchoring systems composed of rigidly interconnected circular rings are often used. In such cases, the engineer faces the problem of analyzing a shell composed of rigidly connected rings with different cross-sections. Similar difficulties arise when the rings are made of different materials and the internal force distribution must be investigated.
The direct calculation of such structures using classical coupling techniques is difficult and, in many cases, practically impossible. The problem becomes significantly simpler when one applies the general theory for constructing **discontinuous solutions of ordinary differential equations**, developed by **Sh. E. Mikeladze**.

Main Part

In this paper, based on the theories of Sh. E. Mikeladze, a discontinuous solution of the governing differential equation is constructed. This approach makes it possible to conveniently account for abrupt changes in individual geometric, kinematic, and static characteristics of shells.
We consider a tunnel lining modeled as an open cylindrical shell with stepwise changes in the radius of curvature R and thickness h. The structure is subjected to distributed vertical load p and horizontal load q. The load p acts over the entire span, while q is applied at a certain height, beginning from a specified point $s = s_1$ (see Figure 1).
In addition to external loads, the shell is influenced by the reaction of an elastic medium, which acts only in regions where radial displacements are positive (i.e., directed outward along the shell normal). The shell edges at the beginning of this region are assumed to be elastically clamped. Of the curved edges, one is free while the other is hinged.

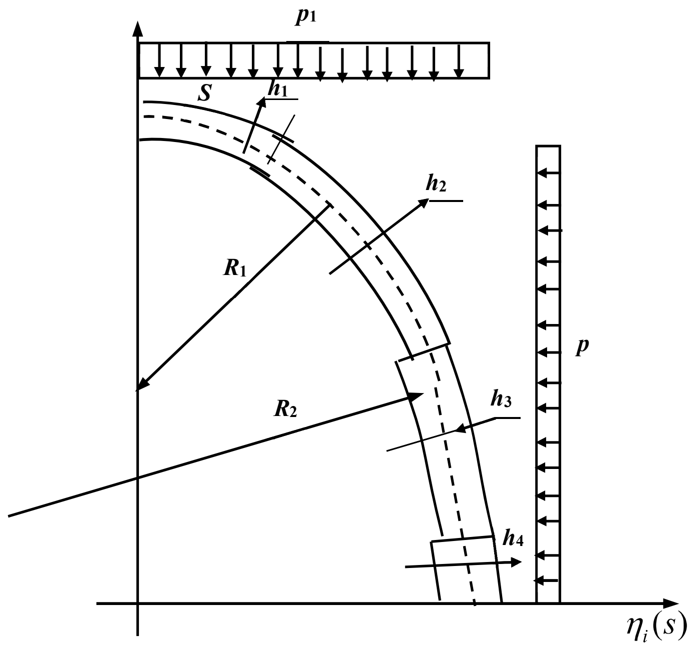

Fig. 1

The calculation of the cylindrical shell is performed using the variational method [1, 2]. The resulting system of governing equations, expressed in the form proposed by V. Z. Vlasov, has the form:

$$\sum_{i=1}^{n_1} a_{ij} F_i^{(4)}(x) + \sum_{i=1}^{n_1} b_{ij} F_i(x) + c_j = 0,\ j = 1,2,3, \dots n_1 \quad (1)$$

$$a_{ij} = E\int h\eta_i(s)\eta_i(s)ds, \qquad 12b_{ij} = E\int R^2 h^3 \eta_i^{(4)}(s)\eta_j^{(4)}(s)ds,$$

$$c_j = -\int q_\theta \eta'_j(s)ds + \int R q_n \eta_j^{(2)}(s)ds$$

Where:

- E denotes the modulus of elasticity,
- the normal and tangential components of external loads are q_n and q_θ
- the functions $F_i(x)$ and $\eta_i(s)$ form a displacement function

The displacement function is represented as:

$$\Phi(x,s) = \sum_{i=1}^{n_1} F_i(x)\eta_i(s),$$

This function satisfies the prescribed boundary conditions. The assumed functions are chosen to be linearly independent and may be interpreted as admissible displacement modes, such as the deflection function of a curved beam of unit width corresponding to a transverse strip of the shell.

Assuming:

$n_1 = 1$, equation (1) can be rewritten as

$$F^{(4)}(x) + b_0 F(x) = c_0 \quad (2)$$

Where:

$$b_0 = b_{11}/a_{11}, \quad c_0 = -c_1/a_{11}.$$

the boundary conditions are written as follows:

- at the free edge:

$$F^{(2)}(o) = F^{(3)}(o) = 0, \quad (3)$$

- at the fixed edge:

$$F(l) = F'(l) = 0 \quad (4)$$

The solution of equation (2), subject to boundary conditions (3) and (4), has the form:

$$F(x) = c_0[1-(\cos\alpha sh\alpha - \sin\alpha ch\alpha)(\sin\beta x ch\beta x + \cos\beta x\, sh\beta x/\gamma +$$
$$+2\cos\alpha ch\alpha\cos\beta x\, ch\beta x/\gamma\,]/b_0\,.$$

Where

$$\alpha = \beta l\,,\ \beta = \sqrt[4]{b_0/4}\,,\ \gamma = \cos^2\alpha sh^2\alpha - \sin^2\alpha ch^2\alpha - 2\cos^2\alpha ch^2\alpha\,.$$

Thus, the remaining task is to select an appropriate displacement function $\eta_1(s)$. To accomplish this, we examine the bending of a curved beam with piecewise-constant thickness and radius of curvature.

Governing Equation of the Curved Beam

The elastic equilibrium equation is written as:

$$w^{(5)}(s) + Aw^{(3)}(s) + Bw'(s) = f(s)\,, \qquad (5)$$

Where:

$$A = 2/R^2\,,\ \ B = 12\,K/Eh^3 + 1/R^4\,,$$
$$f(s) = 24[p\sin(s/R + \sum\varphi_i) + p_1\cos(s/R + \sum\varphi_i)/ERh^3\,,$$
$$\varphi_i = 0 \ \text{ at } s \le s_i \ \text{ and } \varphi_l = s_i\left(1/R_i - 1/R_{i+1}\right) \ \text{ at } s_l < s < s_{i+1}\,.$$

- w denotes the deflection of the curved beam w denotes the deflection of the curved beam
- $s_l -$ is the ordinate of the point at which the radius changes abruptly, and K- is the coefficient of the Winkler elastic medium.

The annular strain is assumed to be zero, leading to the differential equation governing the tangential displacement v:

$v'(s) + w/R = 0\,.$

The boundary conditions are as follows:

- at the axis of symmetry (s = 0): $w' = w^{(3)} = v = 0$;
- at the supports $(s = s_0): w = v = 0$ $w' = E\cdot w^{(2)}/K\,.$

The stepwise variation of radius of curvature, thickness, load intensity, and the foundation coefficient K leads to jumps of the first kind in the derivatives of the deflection function w. Consequently, it is necessary to construct discontinuous solutions of equation (5). Following Sh. E. Mikeladze, the solution is represented in the form of a generalized w:

$$w(s) = w(0) + \sum_{k=1}^{n}\left\{\left[w'(0)d^{(k)} + w^{(3)}(0)b^{(k)}\right]\sin\left(\frac{k\pi}{2}\right) + \left[w^{(2)}(0)d^{(k)} + w^{(4)}(0)b^{k}\right]cos\frac{k\pi}{2}\right\}\frac{s^k}{k!} +$$
$$\sum_{i=1}^{m}\sum_{k=0}^{n}\delta_{ik}\frac{(s-s_i)^k}{k!}H(s-s_i) \qquad (6)$$

Where:

H(ξ)=0 for $\xi \le 0$ and H(ξ)=1 $\xi > 0$

$$b^{(k)} = -A(0)b^{(k-2)} - B(0)b^{(k-4)}\,,\quad d^{(k)} = -A(0)d^{(k-2)} - B(0)d^{(k-4)}\,,$$
$$c^{(k)} = -A(0)c^{(k-2)} - B(0)c^{(k-4)} + f^{(k-5)}(0) \ \text{ for } k \ge 5\,,$$
$$\text{and}\quad b^{(1)} = b^{(2)} = d^{(3)} = d^{(4)} = 0\,,\ b^{(3)} = b^{(4)} = d^{(1)} = d^{(2)} = 1 \ \text{ and } c^{(1)} = c^{(2)} = c^{(3)} = c^{(4)} = 0$$

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and the jump magnitudes at points of discontinuity are determined from continuity conditions on deflection, rotation angle, bending moment, shear force, and normal force.

The jumps δ_{ik} at $k \le 4$ are determined from the condition of continuity of the deflection, angle of rotation, bending moment, shear force and normal force, and at $k \ge 5$ – according to the relation obtained by differentiating (5) the appropriate number of times.

Determination of Initial Parameters

To determine the initial parameters $w^{(k)}(0)$ (k = 0, 1, 2, 3, 4), the method of successive approximations is used. The essence of this method is as follows: in the zeroth approximation, δ_{ik} is assumed to be zero; based on the boundary conditions, the parameters $w^{(k)}(0)$ are determined, and then δ_{ik} in subsequent approximations, the procedure is repeated with new values (δ_{ik} And so on, until the required accuracy is achieved.)

After constructing solution (6), we return to selecting the function $\eta_1(s)$.

In accordance with the variational method, we assume that:

$\eta_1^{(2)}(s) = w/R$. Then $\eta_1'(s) = \int_0^s (w/R)ds + \zeta_1$ and

$$\eta_1(s) = \int_0^s (s-t)(w/R)dt + \zeta_1 s + \zeta_2 .$$

The constants ζ_1 and are ζ_2 determined from the boundary conditions on the straight edges.

Reference

A detailed analysis is carried out for the case where the beam thickness changes abruptly at three points: $(s = s_1,\ s_3,\ s_4)$

In this case, the point of load discontinuity p_1 coincides with s_1, and the point of curvature radius discontinuity coincides with s_3.

Example parameters:

uniform vertical load p = 25 t/m^2, additional distributed load $p_1 = 1{,}5$ t/m^2, elastic modulus $E = 3,21 \cdot 10^6$ t/m^2. The shell thicknesses were taken as $h_1 = 0{,}7$ m, $h_2 = 0{,}8$ m, $h_3 = 0{,}95$ m, $h_4 = 1{,}1$ m, The characteristic arc-length coordinates of thickness discontinuities are: $s_1 = 2$ m, $s_3 = 5{,}5$ m, $s_4 = 7$ m, $s_0 = 8$ m, with the total span length l = 8 m, The radii of curvature were assumed as R_1 = 5,81 m, R_2 = 8, 5 m, The Winkler elastic foundation coefficient was taken as $K = 16 \cdot 10^3$ t/m^2.

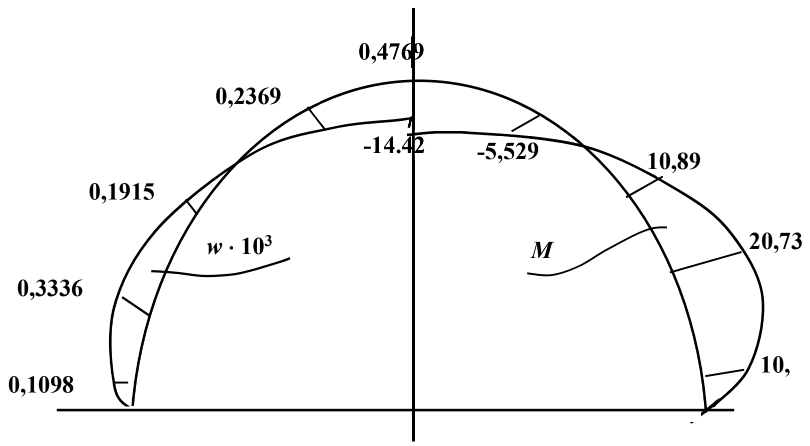

Fig. 2

Figure 2 illustrates the deflection and bending-moment diagrams for the free curved edge of the shell.

Conclusion

In this study, a tunnel lining of oval cross-section with stepwise variations in thickness and radius of curvature was analyzed using the variational method. The governing differential equation of the shell was solved by constructing a discontinuous solution based on the theory developed by Sh. E. Mikeladze, which makes it possible to explicitly account for abrupt changes in geometric and mechanical parameters.

The proposed approach allows the effects of discontinuities in thickness, curvature, loading intensity, and elastic foundation stiffness to be incorporated into the analytical solution without introducing artificial compatibility conditions or complex coupling procedures. The method ensures continuity of displacement, rotation, bending moment, shear force, and normal force at the points of parameter variation, thereby preserving the mechanical consistency of the model.

A numerical example was presented to illustrate the applicability of the method. The obtained deflection and bending-moment diagrams for the free curved edge demonstrate the influence of stepwise parameter changes on the stress–strain state of the tunnel lining. The results confirm that local stiffness variations significantly affect the internal force distribution and deformation pattern of the shell.

The developed analytical framework provides a practical and efficient tool for the calculation of tunnel linings and similar shell structures with piecewise-constant properties. It can be extended to the analysis of shells composed of different materials or resting on nonuniform elastic foundations, and it may serve as a basis for further refinements in the analytical design of underground structures.

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