Determining the rational thickness of a prefabricated dome made of a finite number of spherical shells

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Abstract The problem of calculating a prefabricated dome, the thickness of which is selected in such a way that the maximum deflection is equal to a predetermined value, is considered.
The geometry of the structures and the nature of the acting load determine the involvement of the mathematical apparatus of constructing discontinuous integrals of differential equations in solving the problem.
An algorithm for solving the problem is constructed, including two nested iterative processes.
The problem is solved both for a prefabricated dome and for a corresponding round slab.
A specific example shows the efficiency in terms of material savings of using flat prefabricated domes, compared to flat slabs
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INTRODUCTION

Circular plates and gently sloping domes are widely used as roof and ceiling structures for industrial and civil buildings. Usually, the calculation of such structures is of a verification nature. For given geometric dimensions, mechanical properties of the structure material and the load acting on it, the stress-strain state of the structure is determined. Then, the extent to which this state satisfies the operational conditions of the structure is checked.
Often, the shape and dimensions of structures are established based on technological, aerodynamic or aesthetic considerations. In such cases, it is natural to look for ways that are different from the classical ones, ensuring compliance with the predetermined conditions of rigidity or strength.
In the work [1], a method of "parametric" optimal design is proposed, according to which the function determines the shape of structures is given with an accuracy of up to n unknown parameters. These parameters are determined from the conditions of strength, rigidity, stability, etc.
Later, the idea of that method was used to design circular plates [2], stretched-curved annular disks [3] and closed cylindrical shells [4]. In these works, the law of variation of the thickness of structures was a problem with an accuracy of two unknown parameters, which were determined from the conditions of rigidity and strength. Namely: the maximum displacement had to be equal to a predetermined value and plastic deformation was formed in the fibers with the maximum stress along the thickness.
In this work, a prefabricated flat dome composed of a finite number of spherical shells is considered, which is subjected along with a uniform load, to the action of concentrated transverse forces distributed

along parallels and bending moments. The thickness of the shell is determined in such a way that the maximum deflection is equal to a predetermined value.

Then, the same problem is solved for a round plate, and in a specific example, by comparing the results, it is shown what advantage, in terms of material savings, a prefabricated flat dome has in relation to the corresponding plate.

1. Prefabricated shallow dome

Let us consider the structure of a dome, which is a shallow shell composed of spherical parts with different radii of curvature. The structure is subject to both a uniformly distributed load of intensity q and concentrated forces and moments acting along the circular boundaries of adjacent structural elements.

Let us select the thickness of the dome in such a way that the rigidity condition is obviously satisfied. Namely, the maximum deflection of the structures must be equal to a predetermined value .

The equilibrium equation of a shallow spherical shell, written in displacements, is known to have the form [5]:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)+\frac{Eh}{R^2S}w=\frac{q}{D}, \qquad (1)$$

where w- is the deflection of the shell, E- is Young's modulus, $D=\frac{Eh^3}{12(1-\nu^2)}$ the cylindrical bending rigidity, h- is the thickness, and R- is a piecewise constant radius, which takes the following values: and . $R=R_1$ $(0\le r\le r_1)$ и $R=R_{k+1}$ $(r_k\le r\le r_{k+1},\ k=1,2,$[redacted]$)$.

Let us move on to a dimensionless quantity $\eta=r/b$ and give equation (1) the following form:

$$\frac{d^2\varphi}{d\eta^2}-\frac{1}{\eta}\frac{d\varphi}{d\eta}-\frac{\varphi}{\eta^2}=\frac{Qb^2}{D}, \qquad (2)$$

Where

$$\varphi=-\frac{dw}{bd\eta},$$

$$Q=\frac{\sum_{k=1}^{n}Q_k\eta_k}{\eta}+\frac{b}{\eta}\int_0^{\eta}\left(q-\frac{Eh}{R^2}w\right)\eta d\eta,$$

a Q_k denotes a vertical load uniformly distributed along the parallels: . $\eta=\eta_k$.

The boundary conditions are as follows:

$\varphi=0$ at $\eta=0$ and $w=0$ $\varphi=0$, at . $\eta=1$

Let us analyze the desired function and its derivatives.

Due to the flatness of the structures, the function w- can be considered continuous. As for the angle of rotation φ, its continuity is ensured by the rigidity of the connection of the individual parts to each other. Bending moments and transverse forces acting along parallels , an abrupt change in φ' and φ'' . The piecewise constant nature of the change in radius R- is the cause of the abrupt change in $\varphi^{(3)}$.

Let us construct a solution to equation (2). For this purpose, we will represent the desired function φ in the form of the Maclaurin formula, generalized by Sh.E. Mikeladze [6]:

$$\varphi=\varphi(o)+\varphi'(o)\eta+\sum_{k=1}^{n}\delta_k(\eta-\eta_k)+\int_0^{\eta}(\eta-t)\varphi''(t)dt, \qquad (3)$$

Where

δ_k denotes the jump of functions $\varphi'(\eta)$ at points $\eta=\eta_k$.

For $\varphi'(\eta)$ we have:

$$\varphi'(\eta)=\varphi'(o)+\sum_{k=1}^{n}\delta_k+\int_0^{\eta}\varphi''(t)dt. \qquad (4)$$

As a result of substituting φ and φ' into (2), we obtain the Volterra integral equation of the second kind:

$$\varphi''(\eta) = F(\eta) - \int_0^\eta \frac{t}{\eta^2} \varphi''(t)dt,$$

Where

$$F(\eta) = -\frac{Qb^2}{D} - \frac{\sum_{k=1}^{n} \delta_k \eta_k}{\eta^2}. \quad (5)$$

The resolvent of the resulting integral equation has the form:

$$. \ \Gamma(\eta, t) = \frac{t^2}{\eta^3}$$

As for the solution, it will be written as follows: .

$$\varphi''(\eta) = F(\eta) - \int_0^\eta \frac{t}{\eta^3} F(t)dt.$$

Having φ'', we can determine φ' and φ, and based on the dependence , $\varphi = -\frac{dw}{bd\eta}$ – the deflection -w:

$$W = w(o) - b\int_0^\eta \varphi d\eta. \quad (6)$$

The constants w =0, φ = 0 and are determined from the boundary conditions.

We have::

$$\varphi(o) = 0,$$

$$\varphi'(o) = -\sum_{k=1}^{n} \delta_k (1 - \eta_k) - \int_0^1 (1-t)\varphi''(t)dt,$$

$$w(o) = b\int_0^1 \varphi d\eta.$$

As for the jumps δ_k, they are known quantities:

$$\delta_k = \frac{M_k}{D},$$

where M_k are the values of the moments acting along the parallels $\eta = \eta_k$.

Note that the right-hand side of equation (2) and, consequently, all relations following from it contain -w as an unknown function, for the determination of which the method of successive approximations is used.

The rational thickness is found by the method of simple iteration according to the formula [7]:

$$h_{i+1} = h_1 + æ[w(o) - w_0].$$

$$(æ = 1 - m), \ m = 1, 2, 3 \ldots$$

Thus, the presented algorithm for determining the rational thickness includes two nested iterative processes, which is very convenient for calculation on a digital computer.

2. Circular plate

Let us consider a circular plate that is subject to the same loads as the gently sloping dome.

Let us determine the displacement of the plate from concentrated bending moments. For this purpose, we can use the relations that were given for the dome. Indeed, assuming in (5) $Q = 0$,we obtain formulas describing the deformed state of a circular plate subject to moments distributed along the circumferences.

Let us consider in detail the case when the load acts only along one circumference $\eta = \eta_1$ and give the corresponding relations.

We will have:

$$\varphi(\eta) = \varphi(o) + \varphi'(o)\eta +$$

$$\delta_1 \left(\frac{\eta_1 \eta^3}{6} - \frac{\eta_1^2 \eta^2}{2} + \frac{\eta_1^3 \eta}{2} - \eta_1 \ln \eta_1 - \frac{\eta_1^4}{6} \right).$$

Satisfying the boundary conditions: $\varphi(o) = \varphi(1) = 0$, we obtain:

$$\varphi'(o) + \delta_1 \left(\frac{\eta_1}{6} - \frac{\eta_1^2}{2} + \frac{\eta_1^3}{2} + \eta_1 \ln \eta_1 - \frac{\eta_1^4}{6} \right).$$

For the deflection w based on (6) we have:

$$w = w(o) - \frac{b\varphi'(o)^2}{2} - b\delta_1 \left[\frac{\eta_1 \eta^4}{24} - \frac{\eta_1^2 \eta^3}{6} + \frac{\eta_1^3 \eta^2}{6} - \right.$$

$$\left. -\eta_1(\eta \ln \eta - \eta) + \eta_1 \eta \ln \eta_1 - \frac{\eta_1^4 \eta}{6} + \frac{\eta_1^5}{24} - \eta_1^2 \right].$$

From the condition w[I] = 0, we obtain:

$$w(o)=\frac{b\varphi'(o)}{2}-b\delta_1$$
$$\left(\frac{25}{24}\eta_1-\frac{7}{6}\eta_1^2+\frac{\eta_1^3}{4}-\frac{\eta_1^4}{6}+\frac{\eta_1^5}{24}+\eta_1\ln\eta_1\right).$$

As for the expressions for displacements from distributed - q and concentrated Q_1 transverse loads, we have for them [8]:

$$w=\frac{qb^2}{64D}(1-\eta^2)+$$
$$\frac{pb^2}{8\pi D}\left[\left(\eta_1^2+\eta^2\right)\ln\eta_1+\frac{(1+\eta^2)(1-\eta_1^2)}{2}\right].$$

At the center of the plate, the deflection from all loads must be equal to a predetermined value w_0:

$$w(o)=\frac{b\varphi'(o)}{2}-b\delta_1$$
$$\left(\frac{25}{24}\eta_1-\frac{7}{6}\eta_1^2+\frac{\eta_1^3}{4}-\frac{\eta_1^4}{6}+\frac{\eta_1^5}{24}+\eta_1\ln\eta_1\right)+$$
$$+\frac{qb^2}{64D}+$$
$$\frac{pb^2}{8\pi D}\left(\eta_1^2\ln\eta_1+\frac{1-\eta_1^2}{2}\right).$$

By specifying specific values: b, η_1 , E, , q and Q_1, we can determine the thickness of the structures that ensures compliance with the predetermined rigidity condition.

3. Numerical results and conclusions

Let us consider a specific structure in the form of a prefabricated dome composed of two spherical parts. Let us assume that q = 500 kg/m^2, Q_1= 200 kg/m, M_1 = 100 kgm/m, η_1= 0.5, w_0 = 1•10–4 m, R_1 = 20 m, R_2 = 10 m, b = 2 m, E = 1•1010 kg/m^2, æ = 10, v = 0.

For the numerical implementation of the task, a program was compiled for calculation on a digital computer.

The thickness corresponding to the specified rigidity condition turned out to be equal to 0.055 m.

For a flat plate, the thickness is 0.104 m.

Analysis of the obtained results allows us to conclude that even with an insignificant lifting arrow (of the order of the thickness), the rigidity of the prefabricated dome is significantly greater than the rigidity of the plate. And when it is necessary to design a structure with a predetermined rigidity condition, a flat prefabricated dome provides significant savings in material. In the case under consideration, the thickness of the slab exceeds ≈1.9 times the thickness of the dome.

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