Linear Dependence of Pressure on Volume in Some Ideal Gas Problems

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Abstract

The issues of changing the amount of heat and changing the molar heat capacity in problems of nonmonotonic thermodynamics are considered. Dependence of the ideal gas pressure on volume in the form of a linear function. With a negative slope. The gas molar heat capacity.

Keywords: Nonmonotonic Thermodynamics, heat, molar heat capacity

Problem situation: One mole of a monoatomic ideal gas has expanded in the process $1 \rightarrow 2$, in which the gas pressure depends linearly on its volume, as shown in Figure 1.

   1) What is the total amount of heat absorbed by the gas in the process $1 \rightarrow 2$?

   2) What amount of heat was transferred to the gas and the quantity of heat given off by the gas in the process $1 \rightarrow 2$?

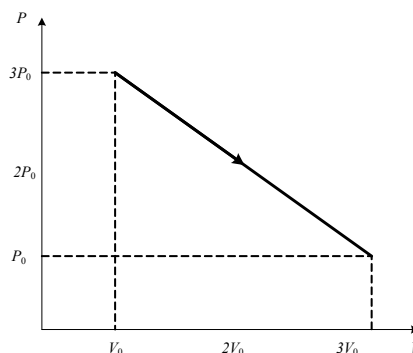   3) Investigate the dependence of a gas's molar heat capacity on the volume of gas.

Fig. 1

The solution of the problem:

   1) The first question is easy to answer: it suffices to use the first law of thermodynamics

$$Q_{12} = \Delta U_{12} + A_{12},$$

where Q_{12} is a total amount of heat absorbed by the gas.

The change of the internal energy of the gas is equal to zero.

$$\Delta U_{12} = U_2 - U_1 = \frac{3}{2} \nu R T_2 - \frac{3}{2} \nu R T_1 = 0$$

Work done by the gas is equal to the area of a figure limited by the graph with a "+" sign:

$$A_{12} = S_{\text{ვაონ.}} = \frac{1}{2}(P_0 + 3P_0)(3V_0 - V_0) = 4P_0V_0 \quad Q_{12} = 4P_0 V_0$$

2) Let us determine the equation of a straight line of the process 1-2 given on the graph: i.e. the relationship between the pressure and volume of gas.
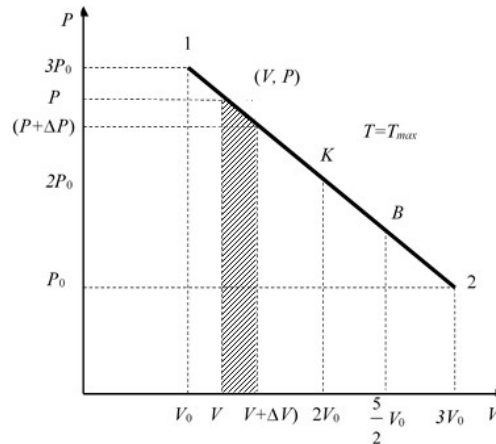

Fig. 2

The equation of a straight line can be determined in different ways:

We obtain $P = P(V) = \dfrac{-P_0}{V_0}V + 4P_0$, (1)

Determine how the gas temperature changes during the process 1-2, as a result of volume changes: That is, determine the dependence $T=T(V)$.

According to the Mendeleev-Clapeyron equation:

Let us substitute $PV=\nu RT$, to the $P=P(V)$ expression

$$T = \frac{PV}{\nu R} = \frac{V}{\nu R}\left(-\frac{P_0}{V_0}V + 4P_0\right)$$

$$T = T(V) = \frac{-P_0}{\nu R V_0}V^2 + \frac{4P_0}{\nu R}V, \qquad (2)$$

The relationship between the absolute temperature and volume of the monoatomic gas has the form of a quadratic function.

We obtained that for $V=2V_0$, the gas temperature reaches its maximum value: by inserting this value into the equation (2), we shall obtain:

$$T_{\max} = \frac{4P_0V_0}{\nu R}$$

By the extremum condition:

$$\frac{dT}{dV} = 0$$

$$\frac{d}{dV}\left(-\frac{P_0}{\nu RV_0} \times V^2 + \frac{4P_0}{\nu R} \times V\right) = 0$$

$$-\frac{P_0}{\nu RV_0} \times 2V + \frac{4P_0}{\nu R} = 0 \quad \Rightarrow \quad V = 2V_0$$

$$\frac{d^2T}{dV^2}. = \frac{d}{dV}\left(\frac{-P_0}{\nu RV_0} \times 2V\right) = \frac{-P_0}{\nu RV_0} \times 2 < 0$$

That is, $T = T(V)$ is a function at a point

$$V = 2\,V_0.$$

It reaches a maximum: $T_{\max} = \dfrac{4P_0V_0}{\nu R}$

Select a point on the graph whose coordinates are ($V{:}P$). Consider an infinitesimal process from that point (Fig. 2), that is, let us increase the volume by $\Delta V(V{+}\Delta V)$: pressure will take on the value of ($P{+}\Delta P$), while the temperature will take on the value of ($T{+}\Delta T$). Let us determine how much heat was absorbed of given off by the gas in this infinitesimal process:

For this process:

$$\delta Q = dU + \delta A$$

The difference in designations of small quantities (δQ, dU, δA) reflects their different mathematical meanings.

The dU quantity is a differential of state function of the internal energy of this system: it is known that it is the so-called total differential, it can be written as the sum of the partial differentials. For example, if the volume and temperature change by the values of dV and dT, then the change of the internal energy can be written as follows:

$$dU = \left(\frac{\partial U}{\partial V}\right)_T \times dU + \left(\frac{\partial U}{\partial T}\right)_V \times dT$$

Work and the amount of heat are not the state functions. These quantities are functions of the process, so their small increments cannot be considered to be the differentials.

If the changes in the volume and temperature are dV are dT respectively, then we still don't know what δQ and δA vary. As a result of the analysis of the process, we will obtain different quantities: if the volume changes first and then the temperature or vice versa, the temperature first and then the volume. None of the obtained values will match the results of a process where volume and temperature change simultaneously. It should be noted that only the internal energy will change in the same way in all three cases. That is what was about the designations.

For simplicity, let us use Δ designations:

$$\delta Q = \Delta U + A$$

Since the process is infinitesimal, we can assume that $p{=}const$ and $A{=}p\Delta v$
The work done by the gas will be equal to the area of the dashed figure in Figure 2:

$$A = \frac{1}{2}(P{+}P{+}\Delta P) \times \Delta V = \frac{1}{2}(2P{+}\Delta P)\,\Delta V$$

The product of $\Delta P \times \Delta V \to 0$, thus $A{=}P \times \Delta V$
As for the change of the internal energy of the gas:

$$U = \frac{3}{2}\gamma R = \frac{\frac{3}{2}PV}{2PV} \Rightarrow \Delta U = \frac{3}{2}\Delta(PV)$$

$$\Delta(PV) = PV + \Delta PV + P\Delta V + \Delta P\Delta V - PV$$

We should take into account that $\Delta P \cdot \Delta V \to 0$, we shall obtain

$$\Delta(PV) = P\Delta V + V\Delta P$$

The resulting expression is valid if

$$\frac{\Delta P}{P} \ll 1, \ \frac{\Delta V}{V} \ll 1$$

To calculate the amount of heat of the infinitesimal process δQ, we shall obtain the following expression:

$$\delta Q = \frac{3}{2}\Delta(PV) + P\Delta V$$

$$\delta Q = \frac{5}{2} P\Delta V + \frac{3}{2} P\Delta V \qquad (3)$$

Let us determine ΔP:

$$\begin{cases} P + \Delta P = P_0\left(4 - \dfrac{V + \Delta V}{V_0}\right) \\ P = P_0\left(4 - \dfrac{V}{V_0}\right) \end{cases}$$

$$P = P_0\left(4 - \frac{V}{V_0}\right) + \Delta P = P_0\left(4 - \frac{V}{V_0} - \frac{\Delta V}{V_0}\right)$$

$$\Delta P = -\frac{P_0}{V_0} \cdot \Delta V$$

Going in an alternative way:

$$P'(V)\frac{\Delta P}{\Delta V} \quad \Rightarrow \quad \Delta P = P'(V)\Delta V = -\frac{P_0}{V_0}\Delta V$$

By inserting into the expression (3), we shall obtain:

$$\delta Q = 2P_0\left(5 - \frac{2V}{V_0}\right)\Delta V$$

We obtain the dependence of the function of a small portion of the amount of heat on the values of ΔV and V in the considered infinitesimal process. $\Delta V > 0$, P0>0 - for all infinitesimal processes.

Let us determine all the values of the volume V for which $\delta Q > 0$ (heat is transferred to the gas).

$$\delta Q{>}0, \text{ if } \left(5 - \frac{2V}{V0}\right) > 0 \Rightarrow v < 2{,}5v_0$$

That is, there is a certain point B at which:

$$V_B{=}2{,}5 \ V_0 \ \text{და} \ P_B{=}1{,}5 \ P_0$$

From the point to the point B, heat is transferred to the gas, that is from the volume V_0 to the volume 2,5 V_0, the gas absorbs energy:

$$V_0 < V < 2{,}5 \ V_0$$

The gas absorbs energy in amount of $Q_{AB} > O$.

As we have seen above, the gas reaches its maximum temperature at the point $V{=}2V_0$. That is, from this point to point B, the gas still absorbs heat, but the temperature is already decreasing.

Find the V-volume values for which $\delta Q{<}0$, that is, the gas gives off heat.

$$\left(5 - \frac{2V}{V_0}\right) < 0 \quad \Rightarrow \quad V > 2{,}5V_0$$

That is, in the process $B \rightarrow 2$, the gas gives off heat, or in the process $1-B$, the gas absorbs heat in the amount of Q_{1B}, while in the process $B-2$, the gas gave off Q_{B2}.

$$Q_{1B} = \frac{9}{2} P_0 V_0 \text{ - the amount of heat transferred to the gas.}$$

$$Q_{B2} = -\frac{9}{8} P_0 V_0 + \frac{5}{8} P_0 V_0 = -\frac{1}{2} P_0 V_0$$

$Q_{B2} = \frac{1}{2} P_0 V_0$ - the given off amount of heat.

The algebraic sum of these two quantities is precisely Q_{12} - the amount of heat that we obtained as an answer to the first question.

3) Let us investigate how the molar heat capacity of the gas changes during the process 1-2.

Let us consider the infinitesimal processes in the vicinity of which the molar heat capacity C is not changed:

$$C = C(V) = \frac{\delta Q}{\gamma \Delta T} = \frac{2 P_0 (5 - \frac{2V}{V0}) \Delta V}{\gamma \Delta T}$$

According to the explanation:

$$T'(V) = \frac{\Delta T}{\Delta V} \quad \Rightarrow \quad \Delta T = T'(V) \Delta V$$

$$\Delta T) = \left(-\frac{P_0}{\gamma R V_0} 2V + \frac{4 P_0}{\gamma R} \right) \Delta V \quad C = C(V) = \frac{2 P_0 \left(5 - \frac{2V}{V_0} \right) \Delta V}{\gamma \left(-\frac{P_0}{\gamma R V_0} 2V + \frac{4 P_0}{\gamma R} \right) \Delta V}$$

$$C = C(V) = \frac{5 - \frac{2V}{V_0} \Delta V}{2 - \frac{V}{V_0}} \times R \qquad (4)$$

We have obtained the graphical depiction of the relationship between the molar heat capacity and volume.

At point 1 of the graph:

$C(1) = C(V_0) = 3R > 0$

At point 2 of the graph:

$$C(2) = C(3\,V_0) = \frac{5-6}{2-3}R = R > 0$$

The molar heat capacity is zero at point B is equal to zero at point B.

$$C(B) = C\frac{5}{2}V_0 = 0$$

$2\,V_0$ – the heat capacity at point K is infinity

$C(2\,V_0) = \infty$

Let us build $C = C(V)$ – the graph of the function.

The molar heat capacity changes its sign twice.

$$V_0 \le V < 2\,V_0,\ C>0;\ C>0;\ \Delta T>0:\ \delta Q = C\gamma \Delta T > 0$$

- $2V_0 < V < \dfrac{5}{2}V_0,\ C<0;\ \Delta T<0;\ \delta Q = C\gamma \Delta T > 0$

- $\dfrac{5}{2}V_0 < V \le 3V_0;\ C>0;\ \Delta T<0;\ \delta Q = C\gamma \Delta T < 0$

Let us consider the problem that describes a real physical device where the process described above can be implemented.

Problem situation: The lower end of a small diameter vertical pipe is soldered. The length of the tube is 2 L. The upper end of the tube is open. In the lower part of the tube, there is an ideal gas at T_0-temperature, the upper part of the tube is filled with mercury.

1) Determine the equation of dependence of the gas pressure on the volume.

2) Determine the maximum temperature in the process of expelling mercury by gas.

3) Determine the amount of heat absorbed by the gas, which is required to completely expel the mercury.

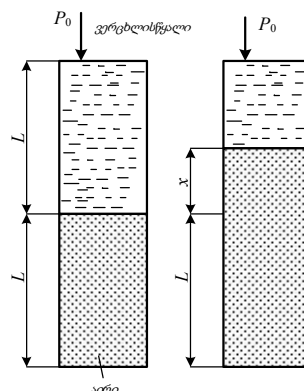An ideal gas ($\gamma = 1$ mole) is monoatomic, the section area of the pipe rate is S. Atmospheric pressure

Fig. 4

The solution of the problem:

Afer the gas is expanded, denote the mole length by X, which is occupied by the gas. In this case, the gas pressure is determined the sum of the pressures generated on the gas by the external atmospheric pressure and the fluid: $P_0 = PgL$ - where P is the density of mercury. The heat pressure is conditioned by the height of pressure $(L-X)$.

Then, according to Pascal's law, the gas pressure will be equal to:

$P = \rho g(2L - X)$

The gas volume: $V = S(L + X)$

$$x = \frac{v}{s} - L = \frac{v}{V_0} L - L$$

We should take into account that $SL = V_0$ – is the initial volume of the gas.

We shall obtain:

$$P = 2\rho g L - \rho g x = 2P_0 - \rho g\left(\frac{V}{S} - L\right)$$

$$P = 2P_0 - \rho g \frac{V}{S} + \rho g L$$

$$P = -\frac{P_0}{V_0} + 3P0$$

The form of the equation coincides with the expression we obtained for the general case: The maximum temperature is reached in the process for the value of $V = \frac{3V_0}{2}$. Using the above resulting expression $T = T(V)$ and taking into account the data of this problem, we shall obtain:

$$T_{max} = \frac{9PoVo}{4vR}$$

$$T_{max} = \frac{9To}{8}; \quad \text{while} \quad Q = \frac{49}{32}PoVo$$

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